Analytical Model of the Rotation of an Artificial Satellite

E. Bois* and J. Kovalevsky†
Observatoire de la Côte d'Azur, Grasse, France

To monitor the rotation of an artificial satellite, using an analytical formulation describing the satellite's attitude in a function of time, may be advantageous in comparison with numerical integration procedures. A second-order theory of the rotation of a satellite was obtained in connection with the very precise attitude determination requested for the scientific exploitation of the European Space Agency astrometric satellite HIPPARCOS. This solution is presented here in the form of self-contained algorithms for the first- and second-order theories with respect to the magnitude of the disturbing torques. It is assumed that the torques can be described in a function of the attitude parameters. The theory was developed with the assumption that the attitude variations consist of a fast rotation around a slowly moving axis. It is valid for a period of time during which the displacement of the rotation axis remains small. This paper presents the algorithms to be used when applying the theory; the actual mathematical developments and proofs have been published elsewhere (see references). Results of new numerical tests are presented; the accuracies obtained by these algorithms are such that in practice the first-order theory is sufficient. The accuracy obtained with the second-order theory is of the order of a few milliseconds of arc in 50,000 s for fast rotating satellites (0.4 rad/s).

I. Introduction

THE attitude of an artificial satellite and its variations with time are some of the main factors to be considered during the operational phase of a space project. Their importance, however, depends very much on the objectives of the mission and on the interference that the rotation parameters of the satellite present with the normal execution of the operations. Two extreme cases may occur:

1) The attitude of the satellite is an ignorable parameter that plays no role. An example would be LAGEOS, a spherical satellite covered with laser retroreflectors, for which the attitude is not even accessible. In this case, of course, the problem does not arise.

2) On the other hand, the satellite is to be pointed on given – and often changing – objectives on the Earth or in the sky. In this case there is an active attitude control onboard, and the knowledge of the attitude at every moment is an intrinsic part of the engineering of the satellite. In a somewhat analogous situation, the satellites are stabilized by gravity gradient torque. In fact, there exist a number of cases for which it is necessary to know the attitude of the satellite in a function of time with some defined accuracy. To do so, it is necessary to have onboard some device capable of determining some attitude parameters and a mathematical model representing the expected variations of the attitude with time. The model can be a simple numerical algorithm processing and interpolating the observational data, but it may be of some interest sometimes to have an algorithm that includes the dynamical behavior of the satellite.

The construction of such a general algorithm was steered by the preparation of the space astrometry mission of the European Space Agency, HIPPARCOS, described by Bouffard[1] and Bonnefoy et al.[2] In this mission, the knowledge of the satellite's attitude at each instant is necessary to a very high accuracy: 1 arc second in real time and, later, 0.1 arc second for the direction of the axis of rotation and a few thousandths of an arc second for the rotation angle between two successive gas jet actuations.[3] For this purpose, it was necessary to study a possible accurate representation of the attitude evolution.[4] The result of this study can be found in Bois,[5,6] who showed that there exists an analytical method that can solve the equations of rotation of an artificial satellite under very general assumptions on the origin of the disturbing torques and on the shape and mass distribution of the satellite. The present paper is intended to give the "recipe" for how to use the algorithms without entering into the mathematical proofs that can be found in the previously cited papers by Bois. The algorithms are presented here in such a way that they can be programmed directly without deriving any intermediate steps or expressions.

Actually, many attempts to construct such a model have been made. For instance, 20 years ago, Crenshaw and Fitzpatrick[7] developed a first-order theory for an axisymmetrical rapidly rotating satellite describing a circular orbit. Hitzl and Breakwell[8] dealt with resonant cases of the perturbations of a triaxial gravity gradient stabilized satellite on an elliptical orbit. Cochran[9] propounded a formulation suitable for any external torque and for any satellite's dynamic shape, but only when the magnitude of the total external torque is small compared to the magnitude of the rotational angular momentum. Then Cochran[10] included the principal effects of the evolution of the orbit of the satellite's center of mass due to the Earth's oblateness. Liu and Fitzpatrick[11] also studied this particular problem. The case of slow rotation has been studied in detail by Stellmacher,[12,13] taking the example of a strongly magnetic satellite, while Donaldson and Jezewski[14] gave general formulations of the perturbed motion about the center of mass. More recently, Van der Ha[15] has presented an approximate analytical solution of the attitude motion under constant body-fixed torques that is valid for any arbitrary inertia parameters.

The present theory is valid under very general conditions. It is suitable for describing the motion of a satellite that has arbitrary dynamic shape and orbital motion for any magnitude of the ratio "disturbing torques/rotational energy" and for any internal or external disturbing forces producing the torques, be they constant, variable, or strongly variable. The existence of an iterative formation law for the coefficients of the solution is the advantage of the present method. Beyond the saving of computation time, instead of an attitude reconstruction by numerical integration, it is then possible to take into account the effects of the usually neglected perturbations.

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*Associate Astronomer, Centre d'Etudes et de Recherches Geodynamiques et Astronomiques.

†Astronomer, Centre d'Etudes et de Recherches Geodynamiques et Astronomiques.

To the best of our knowledge, this paper presents the first results obtained for a second-order theory; but, of course, if a first-order theory is sufficient, the first part of the algorithm presented herein can be used alone.

II. General Assumptions

The basic principle on which this work is based is that with a model of the shape; the mass distribution and the reflective properties of a satellite being known, it is possible to express the torques acting on the satellite as a function of the three angles (α, β, γ) describing the attitude of the satellite in a fixed reference frame and of some fixed or known time variable directions in space. Let us give some examples:

1) The reaction of a gyroscope is a fixed torque in the body-fixed axes, generally constant in intensity.

2) The rotation of a fixed reaction wheel in the satellite has a fixed direction in the satellite and transfers into the fixed axes through the angles α, β, and γ.

3) The torques due to the solar radiation pressure depend on the attitude of the satellite with respect to the sun, the direction of which can be considered either as fixed or depending on one angle slowly and linearly varying with time.

4) The torques due to the gravity gradient produced by the Earth depend on the attitude of the satellite with respect to the position vector from the Earth to the satellite's center of mass. The direction of this position vector is known through the orbital elements of the satellite. These elements can be expressed approximately in terms of angular variables that are linear functions of time.

5) The torques due to the Earth's radiation pressure or aerodynamic torques present the same properties.

From this, it follows that the expression of the resultant torque N acting upon a satellite can be reduced to a function of the three angles α, β, γ describing the attitude and of a certain number of angles having a linear variation with time, $\lambda_i = \lambda_i^0 + n_i t$, such as orbital elements, the direction of the Earth or the sun, etc. Finally one gets

$$N = F(\alpha,\beta,\gamma,\lambda_{i(i=1,n)}) \tag{1}$$

Since all of the parameters are angles, they are 2π-periodic, and it is possible to express each of the components of N in terms of the Fourier series

$$\sum_{j,k,l,i_1\ldots i_n} A_{j,k,l,i_1\ldots i_n} \cos(j\alpha + k\beta + l\gamma + i_1\lambda_1 + \ldots + i_n\lambda_n) + \sum_{j,k,l,i_1\ldots i_n} B_{j,k,l,i_1\ldots i_n} \sin(j\alpha + k\beta + l\gamma + i_1\lambda_1 + \ldots + i_n\lambda_n) \tag{2}$$

The coefficients are numbers that depend only on the properties of the satellite and the coefficients of the development in λ_i of the positions of the disturbing bodies. The significant parts of Eq. (2) can be determined by various analytical or numerical methods. A possible way is to compute N from Eq. (1) for a large selection of values of the angles and then compute numerically the expression in a multiple trigonometric series. Any other harmonic analysis method could be applied.

In practice, a single model will not be valid for a very long time except in some very special cases. In addition, one of the conditions of the algorithm presented here is that it is valid for an interval of time during which the axis of rotation does not move too much. So, during the time of validity of the algorithm, the sun might be considered fixed and the orbital motion may be described using a single time-varying parameter like the mean anomaly. In conclusion, in many cases, Eq. (2) may include no, or only one, argument λ. In what follows, we shall assume that there is no λ varying with time in the torques.

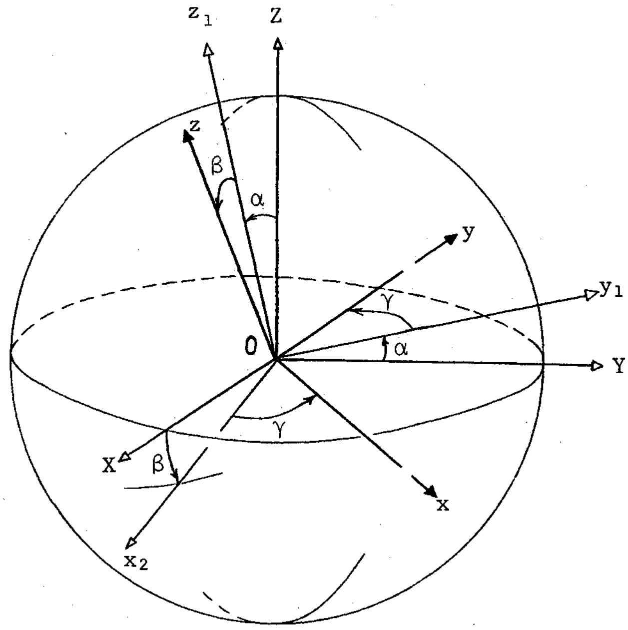

Fig. 1 Definition of the position angles.

III. Equations of Motion

Let O be the center of mass of the satellite. The reference frame with fixed directions is $OXYZ$. Since it is arbitrary, we shall choose the OZ axis close to the rotation axis at the origin of time. Let $Oxyz$ be the body-fixed system of coordinates that will coincide with the principal axes of inertia. The principal centroidal moments of inertia will be called A, B, and C. The angles that will define attitude – namely, the position of $Oxyz$ with respect to $OXYZ$ – are the Tait-Bryan angles defined as follows:

α = first rotation around OX from Y toward Z

β = second rotation around the new Oy_1 axis from z_1 toward X

γ = third rotation around the new $Oz_2 = Oz$ axis from x_2 toward y_1 (see Fig. 1)

The basic assumption that is made in the algorithm is that α and β remain small during the motion, or the part of the motion described by the algorithm. The adopted sequence of angles, composed of two small rotations and one large one (γ), proves to be adequate to express the required motion. Physically, it means that the axis of rotation is close to Oz and that we have indeed chosen OZ also close to it. Finally, as already stated, we assume that no angle λ varying with time is present in N.

The equations are derived from the fundamental equation of the rotation of a solid body:

$$\left(\frac{\mathrm{d}L}{\mathrm{d}t}\right)_{OXYZ} = \left(\frac{\mathrm{d}L}{\mathrm{d}t}\right)_{Oxyz} + \omega \times L = N \tag{3}$$

where

$$L = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \omega$$

$$\omega = \dot{\alpha} + \dot{\beta} + \dot{\gamma}$$

The equations in terms of the Tait-Bryan angles are[5]:

$$(A\cos\beta\cos\gamma)\ddot\alpha + (A\sin\gamma)\ddot\beta + [(-A-B+C)\sin\beta\cos\gamma]\dot\alpha\dot\beta + [(-A+B-C)\cos\beta\sin\gamma]\dot\alpha\dot\gamma + [(A-B+C)\cos\gamma]\dot\beta\dot\gamma + [(B-C)\sin\beta\cos\beta\sin\gamma]\dot\alpha^2 = N_1$$

$$(-B\cos\beta\sin\gamma)\ddot\alpha + (B\sin\gamma)\ddot\beta + [(A+B-C)\sin\beta\sin\gamma]\dot\alpha\dot\beta + [(A-B-C)\cos\beta\cos\gamma]\dot\alpha\dot\gamma + [(A-B-C)\sin\gamma]\dot\beta\dot\gamma + [(A-C)\sin\beta\cos\beta\cos\gamma]\dot\alpha^2 = N_2$$

$$(C\sin\beta)\ddot\alpha + C\ddot\gamma + \{[(-A+B)\cos2\gamma + C]\cos\beta\}\dot\alpha\dot\beta + [(-A+B)\sin\gamma\cos\gamma](\dot\beta^2 - \dot\alpha^2\cos^2\beta) = N_3 \tag{4}$$

where N_1, N_2, and N_3 are the components of N on the three fixed axes OX, OY, and OZ.

We shall assume that the solution may be developed in a power series of coefficient ϵ representing the order of magnitude of the disturbing torques, and write it as

$$\begin{aligned}\alpha &= \alpha_0 + \alpha_1\cdot\epsilon + \alpha_2\cdot\epsilon^2\\ \beta &= \beta_0 + \beta_1\cdot\epsilon + \beta_2\cdot\epsilon^2\\ \gamma &= \gamma_0 + \gamma_1\cdot\epsilon + \gamma_2\cdot\epsilon^2\end{aligned} \tag{5}$$

The starting zeroth-order solution will be taken as follows:

$$\begin{aligned}\alpha_0 &= 0\\ \beta_0 &= 0\\ \gamma_0 &= \omega(t-t_0)\end{aligned} \tag{6}$$

where we have assumed that ω is the nominal rotation velocity of the satellite.

It is always possible to assume $\alpha_0 = \beta_0 = 0$ (see the demonstration on p. 318, Ref. 5). In particular, this is obtained at the origin of time, simply by displacing the reference axes OX, OY, OZ so that they coincide with the initial positions of the rotation axes.

IV. First-Order Solution

The first-order solution[5] is derived from Eq. (4), where the zeroth-order solution, Eq. (6), is substituted in the right-hand members. Their form reduces to

$$N_1 = \epsilon\left[K + \sum_{i=1}^{m}(p_i\sin i\gamma_0 + q_i\cos i\gamma_0)\right] \tag{7a}$$

$$N_2 = \epsilon\left[K' + \sum_{i=1}^{m}(r_i\sin i\gamma_0 + t_i\cos i\gamma_0)\right] \tag{7b}$$

$$N_3 = \epsilon\left[\sum_{i=1}^{m'}(u_i\sin i\gamma_0 + v_i\cos i\gamma_0)\right] \tag{7c}$$

where m and m' are the maximum numbers of terms kept in the developments.

The first-order solution includes linear functions of time and sums of trigonometric terms that are functions only of γ_0:

$$\alpha_1 = n_{\alpha_1}(t-t_0) + \alpha_1^* - \overline{\alpha_1^*} \tag{8a}$$

$$\beta_1 = n_{\beta_1}(t-t_0) + \beta_1^* - \overline{\beta_1^*} \tag{8b}$$

$$\gamma_1 = n_{\gamma_1}(t-t_0) + \gamma_1^* - \overline{\gamma_1^*} \tag{8c}$$

with

$$\alpha_1^* = \sum_{k=1}^{m+1}\frac{a_k^1\sin k\gamma_0 + b_k^1\cos k\gamma_0}{k\omega} \tag{9a}$$

$$\beta_1^* = \sum_{k=1}^{m+1}\frac{c_k^1\sin k\gamma_0 + d_k^1\cos k\gamma_0}{k\omega} \tag{9b}$$

$$\gamma_1^* = \sum_{k=1}^{m'}\frac{e_k^1\sin k\gamma_0 + f_k^1\cos k\gamma_0}{k\omega} \tag{9c}$$

where $\overline{\alpha_1^*}$, $\overline{\beta_1^*}$, and $\overline{\gamma_1^*}$ are the values that α_1^*, β_1^*, and γ_1^* take at time $t = t_0$.

A. Solution in α_1 and β_1

$$n_{\alpha_1} = S_1(p_1, t_1) = -(p_1+t_1)/2\omega C \tag{10a}$$

$$n_{\beta_1} = -D_1(q_1, r_1) = (q_1 - r_1)/2\omega C \tag{10b}$$

The first coefficients ($k = 1$) are given by

$$a_1^1 = S_2(p_2,t_2) + K'/[\omega(A-C)] \tag{11a}$$

$$b_1^1 = D_2(q_2,r_2) + K/[\omega(C-B)] \tag{11b}$$

$$c_1^1 = -D_2(q_2,r_2) + K/[\omega(C-B)] \tag{11c}$$

$$d_1^1 = S_2(p_2,t_2) + K'/[\omega(A-C)] \tag{11d}$$

The next coefficients can be computed using

$$a_k^1 = S_{k+1}(p_{k+1},t_{k+1}) - D_{-(k-1)}(p_{k-1},t_{k-1}) \tag{12a}$$

$$b_k^1 = -S_{-(k-1)}(q_{k-1},r_{k-1}) + D_{k+1}(q_{k+1},r_{k+1}) \tag{12b}$$

$$c_k^1 = -S_{-(k-1)}(q_{k-1},r_{k-1}) - D_{k+1}(q_{k+1},r_{k+1}) \tag{12c}$$

$$d_k^1 = S_{k+1}(p_{k+1},t_{k+1}) + D_{-(k-1)}(p_{k-1},t_{k-1}) \tag{12d}$$

where the quantities S and D are determined by the following formation law:

$$S_j(x_i,y_i) = \frac{x_i(A+jB-C) + y_i(jA+B-C)}{2\omega[C^2 - BC - AC - (j+1)(j-1)AB]} \tag{13a}$$

$$D_j(x_i,y_i) = \frac{x_i(A+jB-C) - y_i(jA+B-C)}{2\omega[C^2 - BC - AC - (j+1)(j-1)AB]} \tag{13b}$$

where the variables x_i and y_i refer to the parameters p_i, q_i, r_i, t_i for $i \geq 1$, and where $j = i$ or $-i$.

B. Solution in γ_1

The coefficients for $k \geq 1$ are given by

$$e_k^1 = -u_k/k\omega C \tag{14a}$$

$$f_k^1 = -v_k/k\omega C \tag{14b}$$

In the first-order solution, n_{γ_1} is arbitrary and we take $n_{\gamma_1} = 0$. In the second-order solution, it becomes determinable, as will be shown in Sec. VI.

V. Second-Order Solution

The second-order solution[6] is composed of linear functions of time, trigonometric terms, trigonometric terms multiplied by time, and quadratic terms in time. The general form is

$$\alpha_2 = n_{\alpha_2}(t-t_0) + n_{\gamma 1}\alpha_2' + \alpha_2^* - \overline{\alpha_2^*} \tag{15a}$$

$$\beta_2 = n_{\beta_2}(t-t_0) + n_{\gamma 1}\beta_2' + \beta_2^* - \overline{\beta_2^*} \tag{15b}$$

$$\gamma_2 = n_{\gamma_2}(t-t_0) + n_{\gamma 1}\gamma_2' + \gamma_2^* - \overline{\gamma_2^*} + \nu(t-t_0)^2 + \theta^*(t-t_0) - 2\int_{t_0}^{t}\theta^*\,\mathrm{d}t \tag{15c}$$

where r, as for the first-order solution, a bar indicates a value taken at $t = t_0$, and where

$$\alpha_2' = n_{\alpha_2}'(t-t_0) + \mathcal{X}^*(t-t_0) - 2\int_{t_0}^{t}\mathcal{X}^*\,\mathrm{d}t \tag{16a}$$

$$\beta_2' = n_{\beta_2}'(t-t_0) + \mathcal{Y}^*(t-t_0) - 2\int_{t_0}^{t}\mathcal{Y}^*\,\mathrm{d}t \tag{16b}$$

$$\gamma_2' = n_{\gamma_2}'(t-t_0) + \mathcal{Z}^*(t-t_0) - 2\int_{t_0}^{t}\mathcal{Z}^*\,\mathrm{d}t \tag{16c}$$

with the following quantities being given as functions of the first-order solution:

$$\begin{array}{l|l|l} \mathcal{X}^* = \dot{\alpha}_1^*/\omega & n_{\alpha_2}' = -n_{\alpha_1}/\omega & \theta^* = -n_{\beta_1}\alpha_1^* \\ \mathcal{Y}^* = \dot{\beta}_1^*/\omega & n_{\beta_2}' = -n_{\beta_1}/\omega & n_{\gamma 2} = \bar{\theta}^* \\ \mathcal{Z}^* = \dot{\gamma}_1^*/\omega & n_{\gamma_2}' = \overline{\mathcal{Z}^*} & \end{array} \tag{17}$$

To compute the remaining terms of Eqs. (15), let us first define and compute the following quantities:

$$x_i = \omega[(iA - B + C)D_i(q_i,r_i) + (iA + B - C)S_{-i}(q_i,r_i)]$$
$$y_i = \omega[-(iA - B + C)S_i(p_i,t_i) - (iA + B - C)D_{-i}(p_i,t_i)] \tag{18a}$$

$$s_i = \omega[(-A + iB + C)S_i(p_i,t_i) - (A + iB - C)D_{-i}(p_i,t_i)]$$
$$z_i = \omega[(-A + iB + C)D_i(q_i,r_i) - (A + iB - C)S_{-i}(q_i,r_i)] \tag{18b}$$

$$\begin{aligned} x_i' &= (-A + B - C)[S_i(p_i,t_i) + D_{-i}(p_i,t_i)] \\ y_i' &= (-A + B - C)[D_i(q_i,r_i) + S_{-i}(q_i,r_i)] \end{aligned} \tag{19a}$$

$$\begin{aligned} s_i' &= (A - B - C)[S_{-i}(q_i,r_i) - D_i(q_i,r_i)] \\ z_i' &= (A - B - C)[S_i(p_i,t_i) + D_{-i}(p_i,t_i)] \end{aligned} \tag{19b}$$

$$\begin{aligned} \lambda &= -\frac{(B-C)}{(A-C)}K' & \mu &= \frac{(A-B+C)}{\omega(C-B)}K \\ \lambda' &= -\frac{(A-C)}{(C-B)}K & \mu' &= \frac{(A-B-C)}{\omega(A-C)}K' \end{aligned} \tag{20}$$

From these, compute

$$\begin{aligned} p_i' &= -(iq_i + x_i) \\ q_i' &= (ip_i - y_i) \end{aligned} \tag{21a}$$

$$\begin{aligned} r_i' &= -(it_i + s_i) \\ t_i' &= (ir_i - z_i) \end{aligned} \tag{21b}$$

$$\begin{aligned} u_i' &= -iv_i \\ v_i' &= iu_i \end{aligned} \tag{21c}$$

$$\Lambda_1^* = \sum_{i=1}^{m}(p_i'\sin i\gamma_0 + q_i'\cos i\gamma_0) \tag{22a}$$

$$\Lambda_2^* = \sum_{i=1}^{m}(r_i'\sin i\gamma_0 + t_i'\cos i\gamma_0) \tag{22b}$$

$$\Lambda_3^* = \sum_{i=1}^{m'}(u_i'\sin i\gamma_0 + v_i'\cos i\gamma_0) \tag{22c}$$

$$\Pi_1^* = \sum_{i=1}^{m}(x_i'\sin i\gamma_0 + y_i'\cos i\gamma_0) \tag{22d}$$

$$\Pi_2^* = \sum_{i=1}^{m}(s_i'\sin i\gamma_0 + z_i'\cos i\gamma_0) \tag{22e}$$

The following quantities are then computed. They are all trigonometric series in γ_0, the coefficients of which define the quantities H, H', P_i, Q_i, R_i, T_i, U_i, V_i, m_2, and m_2':

$$H + \sum_{i=1}^{m_2}(P_i\sin i\gamma_0 + Q_i\cos i\gamma_0) \equiv (\gamma_1^* - \overline{\gamma_1^*})(\lambda + \Lambda_1^*) + \dot{\gamma}_1^*(\mu - \Pi_1^*) \tag{23}$$

$$H' + \sum_{i=1}^{m_2}(R_i\sin i\gamma_0 + T_i\cos i\gamma_0) \equiv (\gamma_1^* - \overline{\gamma_1^*})(\lambda' + \Lambda_2^*) + \dot{\gamma}_1^*(\mu' - \Pi_2^*) \tag{24}$$

$$\begin{aligned} \kappa + \sum_{i=1}^{m_2'}(U_i\sin i\gamma_0 + V_i\cos i\gamma_0) &\equiv (\gamma_1^* - \overline{\gamma_1^*})(\Lambda_3^*) - C(\beta_1^* - \overline{\beta_1^*})(\ddot{\alpha}_1^*) \\ &- C(n_{\alpha_1}\dot{\beta}_1^* + n_{\beta_1}\dot{\alpha}_1^* + \dot{\alpha}_1^*\dot{\beta}_1^*) \\ &+ (A - B)(\cos 2\gamma_0)(n_{\alpha_1} + \dot{\alpha}_1^*)(n_{\beta_1} + \dot{\beta}_1^*) \\ &- ½(A - B)(\sin 2\gamma_0)\left[(n_{\alpha_1} + \dot{\alpha}_1^*)^2 - (n_{\beta_1} + \dot{\beta}_1^*)^2\right] \end{aligned} \tag{25}$$

A. Solution in α_2 and β_2

The quantities n_{α_2} and n_{β_2} of Eqs. (15) are given by

$$n_{\alpha_2} = S_1(P_1, T_1) = -\frac{(P_1 + T_1)}{2\omega C} \tag{26a}$$

$$n_{\beta_2} = -D_1(Q_1, R_1) = \frac{(Q_1 - R_1)}{2\omega C} \tag{26b}$$

The quantities α_2^* and β_2^* are trigonometric series in γ_0 of the following form:

$$\alpha_2^* = \sum_{k=1}^{m_2+1}\frac{a_k^2\sin k\gamma_0 + b_k^2\cos k\gamma_0}{k\omega} \tag{27a}$$

$$\beta_2^* = \sum_{k=1}^{m_2+1}\frac{c_k^2\sin k\gamma_0 + d_k^2\cos k\gamma_0}{k\omega} \tag{27b}$$

In these expressions, the upper index indicates that it is a second-order solution. The coefficients are computable by a procedure identical to the one used in the first-order theory using the same functions S and D defined by Eqs. (13):

$$a_1^2 = S_2(P_2, T_2) + H'/[\omega(A - C)] \tag{28a}$$

$$b_1^2 = D_2(Q_2, R_2) + H/[\omega(C - B)] \tag{28b}$$

$$c_1^2 = -D_2(Q_2, R_2) + H/[\omega(C - B)] \tag{28c}$$

$$d_1^2 = S_2(P_2, T_2) - H'/[\omega(A - C)] \tag{28d}$$

$$a_k^2 = S_{k+1}(P_{k+1}, T_{k+1}) - D_{-(k-1)}(P_{k-1}, T_{k-1}) \tag{29a}$$

$$b_k^2 = -S_{-(k-1)}(Q_{k-1}, R_{k-1}) + D_{k+1}(Q_{k+1}, R_{k+1}) \tag{29b}$$

$$c_k^2 = -S_{-(k-1)}(Q_{k-1}, R_{k-1}) - D_{k+1}(Q_{k+1}, R_{k+1}) \tag{29c}$$

$$d_k^2 = S_{k+1}(P_{k+1}, T_{k+1}) + D_{-(k-1)}(P_{k-1}, T_{k-1}) \tag{29d}$$

B. Solution in $\gamma 2$

In addition to n_{γ_2} given in Eq. (17), the expression for γ_2^* in Eqs. (15) is given by

$$\gamma_2^* = \sum_{k=1}^{m_2'} \frac{e_k^2 \sin k\gamma_0 + f_k^2 \cos k\gamma_0}{k\omega} \tag{30}$$

where, for $k \geq 1$,

$$e_k^2 = -U_k/k\omega C$$

$$f_k^2 = -V_k/k\omega C$$

and the coefficient of the quadratic term in $t - t_0$ are

$$\nu = \tfrac{1}{2}\left[(\kappa/C) - n_{\alpha_1} n_{\beta_1}\right] \tag{31}$$

where κ is the constant term of the development of Eq. (25). This completes the determination of all terms of Eqs. (15).

VI. Secular Term n_{γ_1}

The secular term n_{γ_1} of the first-order theory can be determined at the end of the second-order theory. It is one of the two solutions of the following equation[6]:

$$\mathcal{A}' X_2 + 2\mathcal{B}' X + \mathcal{C}' = 0 \tag{32}$$

where

$$\mathcal{A}' = \epsilon + \frac{\epsilon^3}{\omega^2}\left[(n_{\alpha_1} + \overline{\alpha_1^*})^2 + (n_{\beta_1} + \overline{\beta_1^*})^2\right]$$

$$\mathcal{B}' = \omega + \epsilon\overline{\gamma_1^*} + \epsilon^2\left\{\overline{\gamma_2^*} - \frac{1}{\omega}\left[(n_{\alpha_1} + \overline{\alpha_1^*})^2 + (n_{\beta_1} + \overline{\beta_1^*})^2\right]\right\}$$

$$- \frac{\epsilon^3}{\omega_2}\left\{(n_{\alpha_1} + \overline{\alpha_1^*})(n_{\alpha_2} + \overline{\alpha_2^*}) + (n_{\beta_1} + \overline{\beta_1^*})(n_{\beta_2} + \overline{\beta_2^*})\right\}$$

$$\mathcal{C}' = 2\omega\overline{\gamma_1^*} + \epsilon\left\{(\overline{\gamma_1^*})^2 + 2\omega\overline{\gamma_2^*} + (n_{\alpha_1} + \overline{\alpha_1^*})^2 + (n_{\beta_1} + \overline{\beta_1^*})^2\right\}$$

$$+ 2\epsilon^2\left\{\overline{\gamma_1^*\gamma_2^*} + (n_{\alpha_1} + \overline{\alpha_1^*})(n_{\alpha_2} + \overline{\alpha_2^*}) + (n_{\beta_1} + \overline{\beta_1^*})(n_{\beta_2} + \overline{\beta_2^*})\right\}$$

$$+ \epsilon^3\left\{(\overline{\gamma_2^*})^2 + (n_{\alpha_2} + \overline{\alpha_2^*})^2 + (n_{\beta_2} + \overline{\beta_2^*})^2\right\}$$

Table 1 Maximum differences in seconds of arc between theory and numerical integration

	0.0818 rotation speed, rad/s		0.409 rotation speed, rad/s	
	Integration time, s			
	10^4	5.10^4	10^4	5.10^4
	First-order theory			
$\Delta\alpha$	320.10^{-6}	0.04	3.10^{-6}	330.10^{-6}
$\Delta\beta$	16.10^{-6}	230.10^{-6}	90.10^{-9}	700.10^{-9}
$\Delta\gamma$	0.42	11.0	0.017	0.44
	Second-order theory			
$\Delta\alpha$	340.10^{-6}	0.04	3.10^{-6}	330.10^{-6}
$\Delta\beta$	10.10^{-6}	260.10^{-6}	16.10^{-9}	400.10^{-9}
$\Delta\gamma$	4.10^{-6}	620.10^{-6}	104.10^{-6}	0.0025

VII. General Solution

In conclusion, the general solution is given by adding the solutions of orders 0, 1, and 2, respectively, given by Eqs. (6), (8), and (15) and can be summarized as follows:

$$\alpha = \epsilon[(n_{\alpha_1} + \epsilon n_{\alpha_2})(t - t_0) + \epsilon n_{\gamma_1}\alpha_2' + \alpha_1^* + \epsilon\alpha_2^* - (\overline{\alpha_1^*} + \epsilon\overline{\alpha_2^*})] \tag{33a}$$

$$\beta = \epsilon[(n_{\beta_1} + \epsilon n_{\beta_2})(t - t_0) + \epsilon n_{\gamma_1}\beta_2' + \beta_1^* + \epsilon\beta_2^* - (\overline{\beta_1^*} + \epsilon\overline{\beta_2^*})] \tag{33b}$$

$$\gamma = (\omega + \epsilon n_{\gamma_1} + \epsilon^2 n_{\gamma_2})(t - t_0) + \epsilon[\epsilon n_{\gamma_1}\gamma_2' + \gamma_1^* + \epsilon\gamma_2^* - (\overline{\gamma_1^*} + \epsilon\overline{\gamma_2^*})] + \epsilon^2\left[\nu(t - t_0)^2 + \theta^*(t - t_0) - 2\int_{t_0}^{t} \theta^* \, dt\right] \tag{33c}$$

VIII. Accuracy of the Solution

A number of tests on the accuracy of the first- and second-order solutions presented here have been made. As is easily seen from Eqs. (9), (27), and (30), the larger ω is, the faster the convergence of the series, and the fewer terms one has to keep. For rapidly rotating satellites, the accuracy is limited by the representation of the torques rather than by the theory. In most practical cases, the first-order theory is sufficient.

We have run tests using the expressions of torques and moments of inertia of the satellite HIPPARCOS in a geostationary orbit, and we have compared the analytical solution with a numerical integration of the equations. The torques are of the order of 10^{-5} Nm, while the moments of inertia were taken as $A = 375$ kg m^2, $B = 420$ kg m^2, and $C = 344$ kg m^2. Their final values will be known to about 1% uncertainty. Table 1 gives some significant results that have been obtained and presents the maximum deviation, expressed in seconds of arc, of the literal solution with respect to the numerical integration for two rotation periods.

In the first-order solution the direction of the axis of rotation is better determined than the rotation angle γ itself by several orders of magnitude. The main reason is that there is no determination of the secular term in γ in the first-order solution. When it is possible, in some applications, this secular error can be minimized numerically. The main advantage of the second-order solution is to improve by several orders of magnitude the accuracy in γ. In the results (in particular $\Delta\gamma = 104.10^{-6}$ and 0.0025) obtained for $\omega = 0.409$ and computed with the second-order solution, a drift in the numerical integration seems to occur when the round-off errors become larger than the differences. So, for this high rotation velocity, these results should probably be better with a better integrator.

As remarked previously, the actual limitation will come from the modeling of the torques, which vary significantly, e.g., with the solar activity. The same would apply for aerodynamic torques. This problem also arises with the usual numerical integration method. Therefore, the method presented here has the same limitations, while it is more efficient—computationwise—by at least two or three orders of magnitude.

IX. Conclusion

In this paper we describe an analytical representation of the attitude motion of an artificial satellite. In accordance with the accuracy required in practice, two solutions stemming, respectively, from the first- and the second-order theories are presented. These solutions are given here in the form of self-contained algorithms with directions for their use. Tests and numerical results are also presented.

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